HW#1

1.

$$P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$$
$$= P(E_1)\frac{P(E_1E_2)}{P(E_1)}\frac{P(E_1E_2E_3)}{P(E_1E_2)}\cdots\frac{P(E_1\cdots E_n)}{P(E_1\cdots E_{n-1})}$$
$$= P(E_1\cdots E_n).$$

2.

$$P(E_1) = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13} = \frac{39\cdot 38\cdot 37}{51\cdot 50\cdot 49}.$$
$$P(E_2|E_1) = \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13} = \frac{26\cdot 25}{38\cdot 37}.$$
$$P(E_3|E_1E_2) = \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13} = 13/25.$$
$$P(E_4|E_1E_2E_3) = 1.$$
$$P(E_1E_2E_3E_4) = \frac{39\cdot 26\cdot 13}{51\cdot 50\cdot 49}.$$

3.

(a) $P(E|F) = 0.$

(b) $P(E|F) = P(EF)/P(F) = P(E)/P(F) \geq P(E) = .6.$

(c) $P(E|F) = P(EF)/P(F) = P(F)/P(F) = 1.$

4.

$$c\int_0^2 \left(4x - 2x^2\right)dx = 1$$
$$c(2x^2 - 2x^3/3) = 1$$
$$8c/3 = 1$$
$$c = \frac{3}{8}.$$
$$P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = \frac{3}{8}\int_{1/2}^{3/2}\left(4x - 2x^2\right)dx$$
$$= \frac{11}{16}.$$

5.

$$
\begin{aligned}
P\{M \le x\} &= P\{\max(X_1, \ldots, X_n) \le x\} \\
&= P\{X_1 \le x, \ldots, X_n \le x\} \\
&= \prod_{i=1}^{n} P\{X_i \le x\} \\
&= x^n.
\end{aligned}
$$

$$
f_M(x) = \frac{d}{dx} P\{M \le x\} = nx^{n-1}.
$$

6.

$$
E[e^{tX}] = \int_0^1 e^{tx}\,dx = \frac{e^t - 1}{t}
$$

$$
\frac{d}{dt} E[e^{tX}] = \frac{te^t - e^t + 1}{t^2}
$$

$$
\begin{aligned}
\frac{d^2}{dt^2} E[e^{tX}] &= \frac{[t^2(te^2 + e^t - e^t) - 2t(te^t - e^t + 1)]}{t^4} \\
&= \frac{t^2 e^t - 2(te^t - e^t + 1)}{t^3}.
\end{aligned}
$$

To evaluate at $t = 0$, we must apply l'Hospital's rule.

This yields

$$
E[X] = \lim_{t=0} \frac{te^t + e^t - e^t}{2t} = \lim_{t=0} \frac{e^t}{2} = \frac{1}{2}
$$

$$
\begin{aligned}
E[X^2] &= \lim_{t=0} \frac{2te^t + t^2 e^t - 2te^t - 2e^t + 2e^t}{3t^2} \\
&= \lim_{t=0} \frac{e^t}{3} = \frac{1}{3}.
\end{aligned}
$$

Hence, $Var(X) = \frac{1}{3} - \left[\frac{1}{2}\right]^2 = \frac{1}{12}$.