HW#2

1.

Let T_i denote the time between the $(i-1)^{th}$ and the i^{th} failure. Then the T_i are independent with T_i being exponential with rate $(101-i)/200$. Thus,

$$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{101-i}$$

$$Var(T) = \sum_{i=1}^{5} Var(T_i) = \sum_{i=1}^{5} \frac{(200)^2}{(101-i)^2}$$

2.

For both parts, condition on which item fails first.

(a) $$\sum_{i\neq 1} \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} \frac{\lambda_1}{\sum_{j\neq i} \lambda_j}$$

(b) $$\frac{1}{\sum_{i=1}^{n} \lambda_j} + \sum_{i=1}^{n} \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} \frac{1}{\sum_{j\neq i} \lambda_j}$$

3.

Condition on whether the 1 PM appointment is still with the doctor at 1:30, and use the fact that if she or he is then the remaining time spent is exponential with mean 30. This gives

$$\begin{aligned} &E[\text{time spent in office}] \\ &\quad = 30(1 - e^{-30/30}) + (30+30)e^{-30/30} \\ &\quad = 30 + 30e^{-1} \end{aligned}$$

4.

(a) $$\frac{\lambda}{\lambda + \mu_A}$$

(b) $$\frac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_B}$$

5.

(a) $E[S_4] = 4/\lambda$.

(b) $E[S_4|N(1) = 2]$
$= 1 + E[\text{time for 2 more events}] = 1 + 2/\lambda$.

(c) $E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)]$
$= 2\lambda$.

The first equality used the independent increments property.

6.

(a) $P\{N(T) - N(s) = 1\} = \lambda(T - s)e^{-\lambda(T-s)}$

(b) Differentiating the expression in part (a) and then setting it equal to 0, gives

$$e^{-\lambda(T-s)} = \lambda(T - s)e^{-\lambda(T-s)}$$

implying that the maximizing value is

$$s = T - 1/\lambda$$

(c) For $s = T - 1/\lambda$, we have that $\lambda(T - s) = 1$ and thus,

$$P\{N(T) - N(s) = 1\} = e^{-1}$$

7.

Let T denote the time until the next train arrives; and so T is uniform on $(0, 1)$. Note that, conditional on T, X is Poisson with mean $7T$.

(a) $E[X] = E[E[X|T]] = E[7T] = 7/2$.

(b) $E[X|T] = 7T$, $Var(X|T) = 7T$. By the conditional variance formula
$Var(X) = 7E[T] + 49Var[T] = 7/2 + 49/12 = 91/12$.