SYST 201
Systems Modeling I

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Unit 5: Higher Order Linear Dynamic Systems

- **Problems modeled with higher order dynamic systems**
- **Solutions to second order linear dynamic systems**
- **Long-term behavior of second order linear dynamic systems**
- **Second-order nonhomogeneous dynamic systems**
- **Higher order dynamic systems**

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Learning Objectives

- Recognize problems that can be modeled with higher order dynamic systems
- Develop a higher order dynamic system model from a problem description
- Find the general and particular solutions for a second or higher order linear dynamic system
- Describe the long-term behavior of a second or higher order linear dynamic system
- Find the general and particular solutions for a second order nonhomogeneous dynamic system with polynomial external input term

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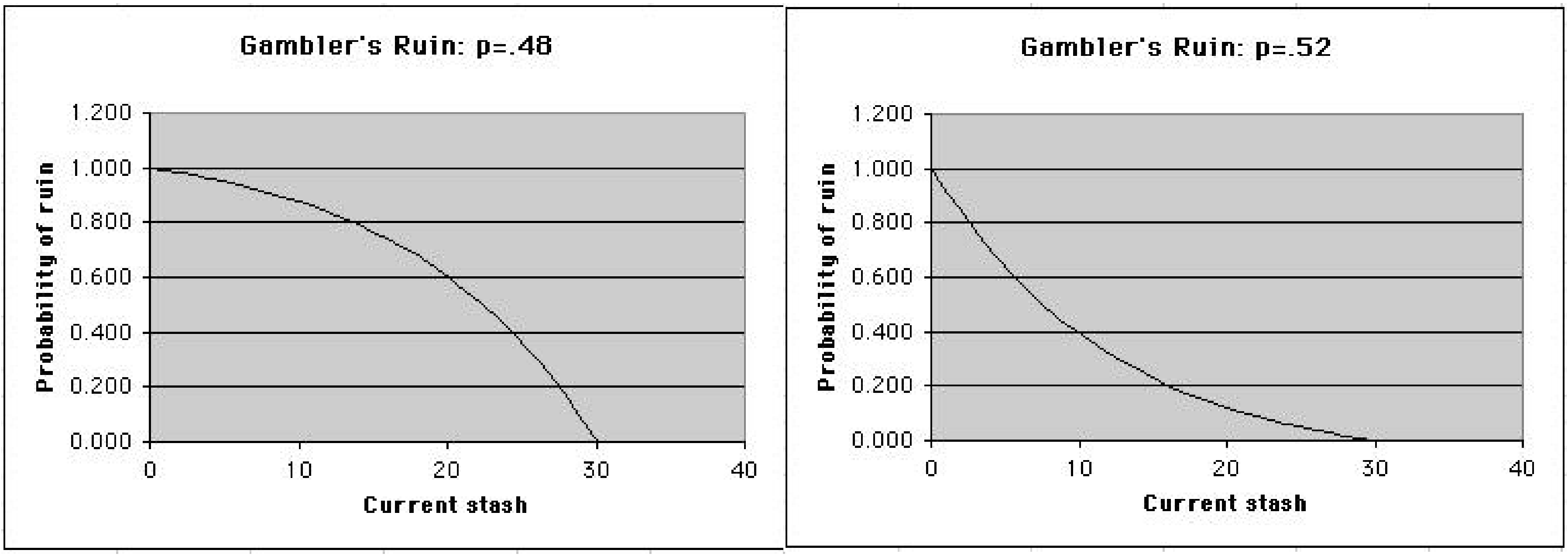
Gambler's Ruin

- **A gambling problem:**
  - **You have a stash of S dollars for betting**
  - **You have a goal of G dollars**
  - **The game:**
    - » **You bet $1**
    - » **You win $1 with probability p (add $1 to your current fortune)**
    - » **You lose $1 with probability 1-p (subtract $1 from your current fortune)**
  - **You keep playing this game till you reach your goal of G dollars or lose your stash of S dollars**
- **Developing a dynamic system model:**
  - **a(n) = probability you will go broke if you have n dollars right now**
  - **There are two ways we could go broke if we have n dollars right now:**
    - » **We could win, have n+1 dollars, and eventually go broke. This has probability p.**
    - » **We could lose, have n-1 dollars, and eventually go broke. This has probability 1-p.**
  - **$a(n) = pa(n+1) + (1-p)a(n-1)$ $\quad a(0) = 1 \quad a(G) = 0$**
- **The gambler's ruin is a second order linear dynamic system**

Gambler's Ruin: Goal G=$30



- **Suppose you start out with $15 and your goal is to double your money**
  - **Compare "subfair game" p=.48 with "superfair game" p=.52**
- **Small difference in the probability of winning make a big difference in your probability of being ruined**
- **This effect is more pronounced the larger your goal**

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Genetics: Sex-linked traits

- **Sex-linked trait (examples: hemophilia, color-blindness)**
  - **Women have 2 X chromosomes and men have 1 X and 1 Y chromosome**
  - **If an allele is on the X chromosome then**
    - » **(A A) and (A a) girls will have the dominant trait and (a a) girls will have the recessive trait**
    - » **(A) boys will have the dominant trait and (a) boys will have the recessive trait**
    - » **(A a) girls have dominant trait but have a 50% chance of passing recessive trait to their sons**
- **Current generation:**
  - **$p(n)$ = proportion of A-alleles and $q(n)$ = proportion of a-alleles among women**
  - **$P(n)$ = proportion of A-alleles and $Q(n)$ = proportion of a-alleles among men**
- **Next generation**
  - **Dominant homozygote women:** $p(n)P(n)$
  - **Heterozygote women:** $p(n)Q(n) + q(n)P(n)$
  - **Recessive heterozygote women** $q(n)Q(n)$
  - **Dominant men** $p(n)$
  - **Recessive men** $q(n)$

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Dynamic Model for Sex Linked Traits

- **Dynamic equation for males**
  - **$P(n+1) = p(n)$**
  - **$Q(n+1) = q(n)$**
- **Proportion of dominant alleles in females in generation n+1**
  - **Equal to the proportion of dominant homozygote females plus half the proportion of heterozygote females**
  - **$p(n+1) = p(n)P(n) + 0.5(p(n)Q(n) + q(n)P(n))$**
- **Proportion of recessive alleles in females in generation n+1**
  - **Equal to the proportion of recessive homozygote females plus half the proportion of heterozgote females**
  - **$q(n+1) = q(n)Q(n) + 0.5(p(n)Q(n) + q(n)P(n))$**
- **Dynamic equation for females**
  - **$p(n+1) = 0.5\, p(n) + 0.5\, P(n) = 0.5\, p(n) + 0.5\, p(n-1)$**
  - **$q(n+1) = 0.5\, q(n) + 0.5\, Q(n) = 0.5\, q(n) + 0.5\, q(n-1)$**
- **This is a second order linear dynamic system**

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Solution to Second Order Linear Dynamic Systems

- **Characteristic equation for a dynamic system**
  - **Dynamic system:** $a(n+2) = b_1\, a(n+1) + b_2\, a(n)$
  - **Characteristic equation:** $x^2 = b_1 x + b_2$
- **The characteristic equation of a second order dynamic system is a second order polynomial**
- **A second order polynomial may have**
  - **2 distinct roots (real or complex)** $x^2 = b_1 x + b_2$
  - **A single double real root** $x^2 = b_1 x + b_2$
- **If the characteristic equation has 2 distinct roots $r \neq s$ then the dynamic system has general solution given by**

  $$a(k) = c_1 r^k + c_2 s^k$$

  and particular solution given by

  $$c_1 = \frac{a(1) - s a(0)}{r - s} \qquad c_2 = \frac{a(1) - r a(0)}{s - r}$$

- **The solution may involve complex numbers but if equation and initial conditions are real then all values are real**

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Equilibrium Points

- **A second order linear dynamic system has an equilibrium point at zero**
- **If both roots are less than 1 in absolute value then the equilibrium value is stable**
- **If either root is greater than 1 in absolute value then the equilibrium value is unstable**

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Example: Fibonacci numbers

- **The Fibonacci series was discovered by the mathematician Fibonacci**
- **The Fibonacci series is a dynamic system in which each term is the sum of the previous two terms:**
    - **1 1 2 3 5 8 13 21 34 …**
    - **a(n+2) = a(n) + a(n+1) a(0) = 1 a(1) = 1**
- **The Fibonacci numbers occur frequently in nature**
    - **Count how many spiral rows of scales there are on a pinecone**
        - » **You may find 8 left winding spirals and 13 right winding spirals**
        - » **You may find 13 left winding spirals and 21 right winding spirals**
        - » **The number of left-winding and right-winding spirals are adjacent Fibonacci numbers**
    - **This phenomenon is called phyllotaxis**
    - **Biologist Paul Green at Stanford argues that phyllotaxis is the simplest self-repeating pattern that can be generated by the growth processes in the growing tips of pinecones, sunflowers, and other organisms that exhibit phyllotaxis**
- **How many ways can a robot arm pick up n objects if it can pick up either one or two objects at at time?**
    - **a(n) is the nth Fibonacci number**

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Roots of a Quadratic Equation

- The quadratic equation $x^2 = b_1 x + b_2$ (equivalently $x^2 - b_1x - b_2$) has roots

$$\frac{b_1 \pm \sqrt{b_1^2 + 4b_2}}{2}$$

- The two roots r and s are obtained by using the positive and negative square roots in this formula
- If $b_1^2+4b_2 < 0$ then the roots are complex numbers
  - Imaginary number: $i^2 = -1$
  - Complex number: $a + bi$
    - Real part is a
    - Imaginary part is b
- Writing the equation in factored form:

$$x^2 - b_1 x - b_2 = 0$$

$$(x-r)(x-s) = 0$$

$$x^2 - (r+s)x + rs = 0$$

$$r + s = b_1 \qquad rs = - b_2$$

- If b_1 and b_2 are real then r+s and rs are also real

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The Fibonacci Numbers

- **We can apply the solution theorem to find the following expression for the nth Fibonacci number:**

$$\mathbf{a(k)} = \frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\left(\frac{\mathbf{1}+\sqrt{\mathbf{5}}}{\mathbf{2}}\right)^{\mathbf{k+1}} - \frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\left(\frac{\mathbf{1}-\sqrt{\mathbf{5}}}{\mathbf{2}}\right)^{\mathbf{k+1}}$$

- **Note that a(n) is always an integer!**
- **This seems like a very complicated formula but it gives us a recipe for direct calculation of Fibonacci numbers**
- **The square root of 5 is 2.236. Inserting this number into the second term of the equation, we see that the second term is always less than 0.5. Therefore, we can find the kth Fibonacci number by computing the first term and rounding to the nearest integer.**

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Characteristic Equation with a Double Root

- **If the second order dynamic system has a characteristic equation with a double root r then the general solution is**

  $$a(k) = (c_1 + c_2 k)\, r^k$$

  and particular solution given by

  $$c_1 = a(0) \qquad c_2 = \frac{a(1) - ra(0)}{r}$$

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Exercise

- **To find the general solution to a second order dynamic system:**
  - **Find the characteristic equation**
  - **Find the roots to the characteristic equation**
  - **Substitute the roots into the solution formula**
- **Find the general solution to the following second-order dynamic systems**
  - **$a(n+2) = -2a(n+1) + 8a(n)$**
  - **$a(n+2) = 16a(n+1) - 64a(n)$**
  - **$a(n+2) = -7a(n+1) - 6a(n)$**
  - **$a(n+2) = 9a(n)$**
  - **$a(n+2) = 2a(n+1) + 15a(n)$**

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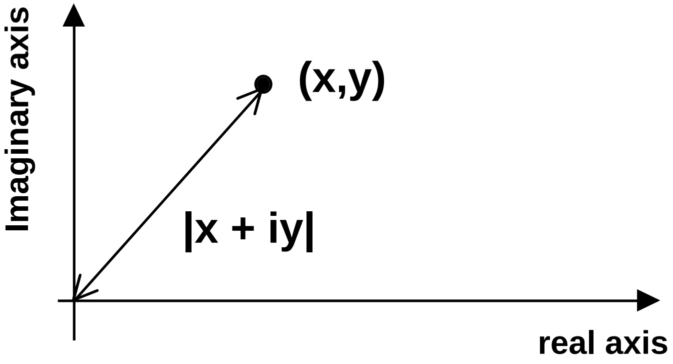
Absolute Value of a Complex Number

- We take tne absolute value of a complex number as follows:

  $$|x+iy| = \sqrt{x^2 + y^2}$$

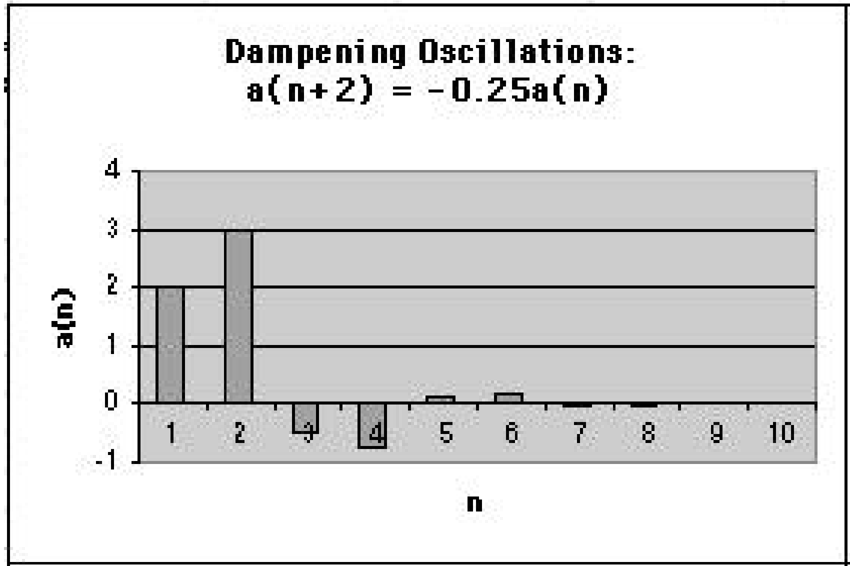
- Both roots of a quadratic equation have the same absolute value
- A little algebra shows that the absolute value of the root is also equal to the square root of the product of the roots
  - Characteristic equation $x^2 = b_1 x + b_2$
  - Roots r and s
  - Absolute value $|r| = |s| = \sqrt{-b_2}$
- *(Note that the textbook changes the sign of b_1 and b_2 from the convention used in the rest of the chapter)*

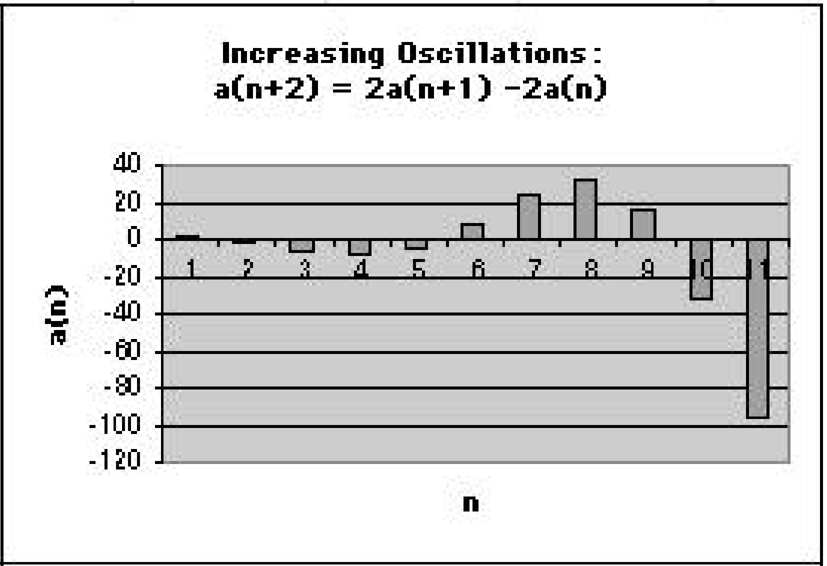


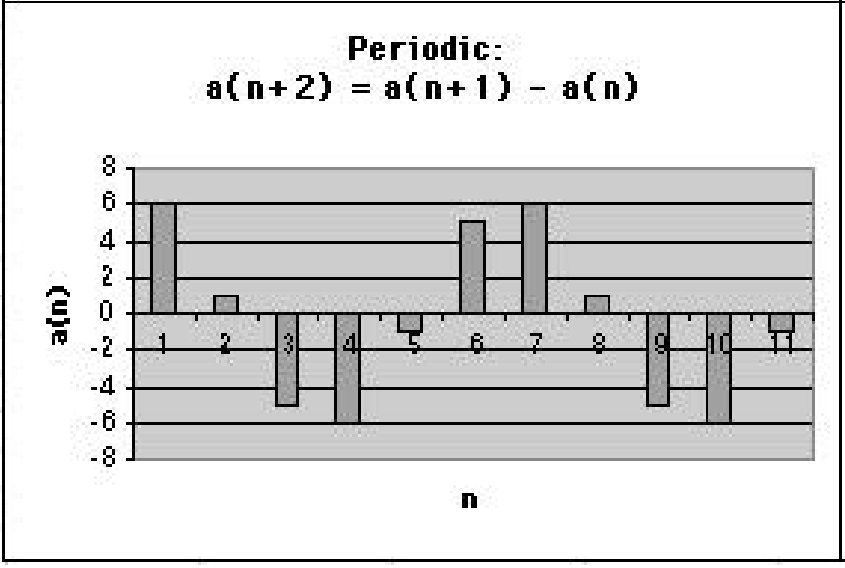
Behavior of a(n) as Function of n

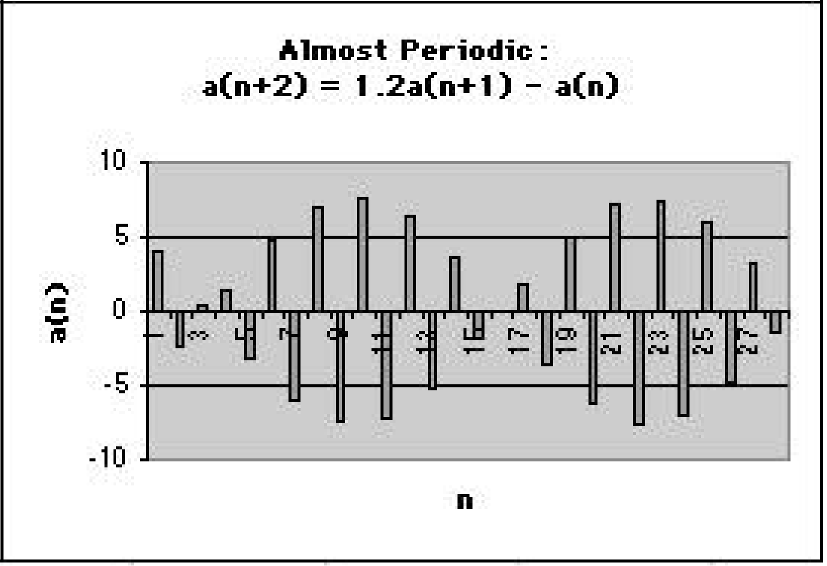
- **Complex roots with absolute value less than 1**
  - **Solution oscillates between positive and negative numbers**
  - **Oscillations dampen toward zero**
- **Complex roots with absolute value greater than 1**
  - **Solution oscillates between positive and negative numbers**
  - **Oscillations increase in amplitude to infinity**
- **Complex roots with absolute value equal to 1**
  - **Solution oscillates between positive and negative values**
  - **Amplitude of oscillations remains constant**
  - **Solution may be periodic**
  - **Solution may be nearly periodic but never repeat itself exactly**

Some Examples


Dampening Oscillations:
$a(n+2) = -0.25a(n)$


Increasing Oscillations:
$a(n+2) = 2a(n+1) - 2a(n)$


Periodic:
$a(n+2) = a(n+1) - a(n)$


Almost Periodic:
$a(n+2) = 1.2a(n+1) - a(n)$

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Long Term Behavior and Stability of Linear Systems

- **System is stable when all roots have absolute value less than 1**
- **System is unstable when at least one root has absolute value greater than 1**
- **Solution oscillates when roots are complex**
    - **Oscillations dampen when absolute value is less than 1**
    - **Oscillations amplify when absolute value is greater than 1**
    - **Oscillations stay at constant amplitude when absolute value is equal to 1**

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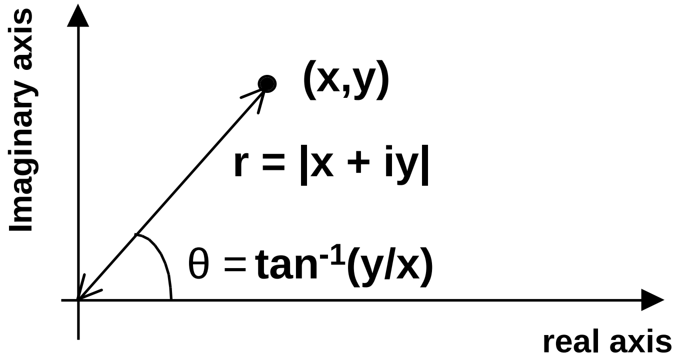
Complex Numbers and Trigonometric Functions

- **The complex number x + iy can be represented as a point in the complex plane**
  - **(x, y) in Cartesian coordinates**
  - **(r, θ) in polar coordinates**
  - $x + i y = r \cos\theta + i r \sin\theta$
- **Some facts about complex numbers**
  - $(\cos\theta + i \sin\theta)^k = \cos k\theta + i \sin k\theta$
  - $(\cos\theta - i \sin\theta)^k = \cos k\theta - i \sin k\theta$
- **Another way of writing the general solution:**
  - $a(k) = r^k(c_3 \cos k\theta + c_4 \sin k\theta)$ $c_3 = c_1+c_2$ $c_4 = i\,(c_1-c_2)$



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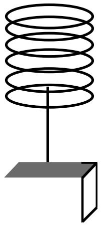
Model of a Vibrating Spring

- **The problem:**
  - **Place a weight on a spring and let it come to rest**
  - **Pull the spring and let go**
  - **The spring will oscillate**
  - **d(n) is the displacement of the spring from its rest position at time n**
  - **Goal: model d(n) as a dynamic system**
- **Some basic physics:**
  - **Hooke's Law: force exerted by spring is proportional to its displacement**
  - **Newton's second law: force is equal to mass times acceleration**
  - **Acceleration is change in velocity divided by change in time**



- **Building the model:**
  - **Average acceleration:** $a(n+1) = \Delta v/\Delta t = v(n+1) - v(n)$
  - **Average velocity:** $v(n+1) = \Delta d/\Delta t = d(n+1) - d(n)$

    $v(n) = d(n) - d(n-1)$
  - **Substitute to find average acceleration** $a(n+1) = d(n+1) - 2d(n) + d(n-1)$
  - **From Hooke's Law and Newton's Law:** $a(n+1) = -pd(n)$
  - **Therefore:** $d(n+1) - 2d(n) + d(n-1) = -pd(n)$

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Comments on the Vibrating Spring Model

- **The dynamic equation:**
  - **$d(n+1) - 2d(n) + d(n-1) = -pd(n)$**
- **This is really a continuous time problem which we have broken into discrete time units**
- **In SYST 202 we will study continuous time systems using the theory of differential equations**
  - **Difference equation: $(d(n+1) - d(n)) - (d(n) - d(n-1)) = -pd(n)$**
  - **Differential equation: $d''(t) = -pd(t)$**
  - **Solutions:**
    - **$d(t) = \sin(p^{1/2}t)$**
      $d'(t) = p^{1/2} \cos(p^{1/2}t)$
      $d''(t) = - p \sin(p^{1/2}t)$
    - **$d(t) = \cos(p^{1/2}t)$**
      $d'(t) = - p^{1/2} \sin(p^{1/2}t)$
      $d''(t) = - p \cos(p^{1/2}t)$

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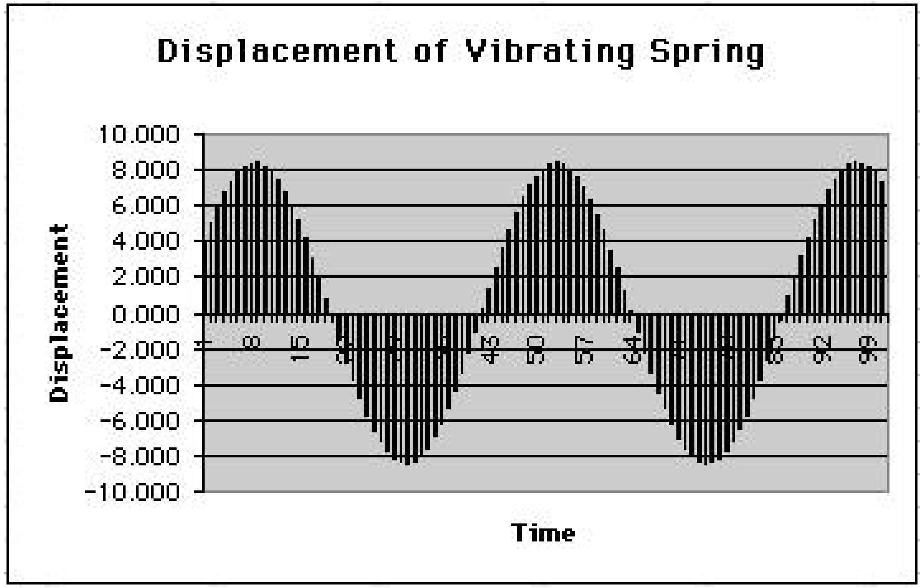
Solution to Vibrating Spring System

- **Assume a small constant of proportionality: p=.02**
- **Dynamic equation: d(n+2) - 1.98d(n+1) + d(n) = 0**
- **Characteristic equation: $d^2 - 1.98d + 1 = 0$**
- **Roots:** $0.99 \pm \sqrt{0.0199}i$
- **The general solution:** $d(k) = c_1\left(0.99 + \sqrt{0.0199}i\right)^k + c_1\left(0.99 - \sqrt{0.0199}i\right)^k$
- **The roots have absolute value equal to 1**
- **The solution oscillates with constant amplitude**
- **Adding a friction term (as discussed in text) dampens oscillations**



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Nonhomogeneous Second Order Systems

- Second order nonhomogeneous systems work similarly to first order nonhomogeneous systems
  - Find the roots of the characteristic polynomial for the corresponding homogeneous dynamic system
  - The general solution has the form of the general solution to the homogeneous dynamic system plus:
    - » A polynomial if the external input term is a polynomial
    - » An exponential if the external input term is exponential
  - There are different cases for
    - » Root(s) equal to 1 if external input term is a polynomial
    - » Root(s) equal to each other if external input term is an exponential
- Find the general solution to the following second-order dynamic systems
  - $a(n+2) = -2a(n+1) + 8a(n) - 3n$
  - $a(n+2) = 16a(n+1) - 64a(n) + 1$
  - $a(n+2) = -7a(n+1) - 6a(n) + 2^n$
  - $a(n+2) = 9a(n) - 6$
  - $a(n+2) = 2a(n+1) + 15a(n) + 5n - 1$
- How do you find the particular solution?

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Rules for Systems with Polynomial Term

- **The general solution a(k) is the sum of two parts:**
  - **Part 1 has the form of the general solution to the corresponding homogeneous dynamic system**
  - **Part 2 is a polynomial term**
- **Part 1**
  - **If roots are different it has the form $c_1r^k + c_2s^k$**
  - **For double roots it has the form $(c_1+c_2k)r^k$**
  - **The coefficients c_1 and c_2 depend on initial conditions**
- **Part 2: where m is the degree of the polynomial term**
  - **If no roots are equal to 1 it is a polynomial of the form $c_3 + c_4k + \ldots + c_{m+3}k^m$**
  - **If one of the roots is equal to 1 it is a polynomial of the form $c_3k + c_4k^2 + \ldots + c_{m+3}k^{m+1}$**
  - **If both of the roots are equal to 1 it is a polynomial of the form $c_3k^2 + c_4k^3 + \ldots + c_{m+3}k^{m+2}$**
  - **The coefficients $c_3, c_4, \ldots, c_{m+3}$ are found by substituting into the dynamic system and solving m+1 equations for m+1 unknowns**
- **Compare these rules with the rules for first-order systems**
- **How would they generalize to higher order systems?**

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Rules for Systems with Exponential Term

- **The general solution a(k) is the sum of two parts:**
  - **Part 1 has the form of the general solution to the corresponding homogeneous dynamic system**
  - **Part 2 is an exponential term b_3p^n**
- **Part 1**
  - **If roots are different it has the form $c_1r^k + c_2s^k$**
  - **For double roots it has the form $(c_1+c_2k)r^k$**
  - **The coefficients c_1 and c_2 depend on initial conditions**
- **Part 2: The form depends on whether p is a root of the characteristic equation**
  - **If none of the roots is equal to p then it is equal to c_3p^k**
  - **If one of the roots r is equal to p then it is equal to $(c_1+c_2k)r^k$**
  - **If both of the roots are equal to p then it is equal to $(c_1+c_2k + c_3k^2)r^k$**
  - **The coefficient is found by substituting into the dynamic system**
- **Compare these rules with the rules for first-order systems**
- **How would they generalize to higher order systems?**

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Gambler’s Ruin: How Long Will Game Last?

- **$e(n)$ = expected number of bets till the game is over if we have n dollars and our goal is N dollars**
- **Boundary conditions: $e(0) = e(N) = 0$**
- **For $0 < n < N$, we will play once, win or lose, and then continue or not depending on whether we have met our goal or lost all our money, or are somewhere in between:**
  - **$e(n) = 1$ + number of additional bets after this trial**
  - **With probability p we win and have n+1 dollars left**
  - **With probability $q = 1-p$ we lose and have n-1 dollars left**
  - **$e(n) = 1 + p\, e(n+1) + q\, e(n-1)$**
- **Example: $N=3$, we start with 1 dollar, and $p=0.48$**
  - **52% chance we lose our dollar and stop because we are broke**
  - **48% chance we win and have \$2**
  - **52% chance we lose and are back to \$1**
  - **48% chance we win and stop at \$3**
  - **$e(1) = 1 + 0.48e(2) + 0.52e(0) = 1 + (0.48)\,[1 + (0.48)e(3) + (0.52)e(1)] + 0.52e(0)$**

    $= 1 + (0.48)\,[1 + (0.48)(0) + (0.52)e(1)] + 0.52(0) = 1.48 + 0.2496e(1)$
  - **$e(1) = 1.97$**

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The Solution

- The general solution for $p \neq 1/2$
  - $e(k) = c_1 + c_2(q/p)^k + k/(q-p)$
- The particular solution for $e(0) = 0$ and $e(N) = 0$
  - $e(0) = c_1 + c_2$
  - $e(N) = c_1 + c_2(q/p)^N + N/(q-p)$
- Solve for c_1 and c_2 and substitute:
  - $$e(k) = \frac{N\left[1-(q/p)^k\right]}{(q-p)\left[(q/p)^N-1\right]} + \frac{k}{q-p}$$
- If $p < 1/2$, and aspiration level is very large (N near infinity)
  - We are almost certain to go broke
  - We expect to make $n/(q-p)$ bets before going broke
- The general solution for $p=1/2$ (fair game)
  - $e(k) = k(N-k)$
  - The expected number of bets goes to infinity as the aspiration level goes to infinity!

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Example: Reproduction

(Problem 1, p. 261)

- **A species has 2 age groups**
  - **Age 0-1**
    - **Survival rate is 0.8**
    - **Reproductive rate is 2 offspring per animal**
  - **Age 1-2**
    - **Survival rate is 0 (no one lives past age 2)**
    - **Reproductive rate is 3.75 offspring per animal**
- **Definitions:**
  - **a(n) = population in age group 0-1 at time n**
  - **b(n) = population in age group 1-2 at time n**
- **The model:**
  - **Animals in age group 1-2 came from age group 0-1 in previous year**
    - **$b(n+1) = 0.8a(n)$**
  - **Animals in age group 0-1 were offspring of someone in previous year**
    - **$a(n+1) = 2a(n) + 3.75b(n)$**
  - **Substitute $b(n) = 0.8a(n-1)$**
    - **$a(n+1) = 2a(n) + (3.75)(0.8)a(n-1) = 2a(n) + 3a(n-1)$**
    - **Roots of characteristic equation are a=3 and a=-1**
    - **General solution is $a(k) = c_1 3^k + c_2(-1)^k$**

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Finding Long Term Population Ratios

- **In this model the equilibrium is unstable and the population increases without bound**
  - **Why won't this really happen?**
- **The long-term limit of the sizes of the two age groups is stable even though the population grows without bound**
  - **$a(k) = c_1 3^k + c_2(-1)^k$**
  - **$a(k)/ c_1 3^k = 1 + (c_2/ c_1)(-1/3)^k$**
  - **The limit as k approaches infinity of $a(k)/ c_1 3^k$ is 1 (*<u>why?</u>*)**
  - **$b(k) = 0.8a(k-1) = 0.8c_1 3^{k-1} + 0.8c_2(-1)^{k-1}$**
  - **$b(k)/ c_1 3^k = (0.8c_1 3^{k-1} + c_2(-1)^{k-1})/ c_1 3^k = 0.8/3 + 0.8/3\, (c_2/ c_1)(-1/3)^k$**
  - **The limit as k approaches infinity of $b(k)/ c_1 3^k$ is 0.8/3 = 0.2667**
  - **The long run ratio of age group 0-1 to age group 1-2 is 1/0.2667 = 3.75**
- **Is it reasonable to believe the model's long run predictions for age group ratios if we don't believe the population size will increase indefinitely? Why or why not?**

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Higher Order Systems

- Dynamic systems of order greater than two can be treated in the same way as second order systems
- To solve a homogeneous dynamic system of any order
  - Find the characteristic polynomial
  - Find the roots of the characteristic polynomial (this may be difficult for higher order polynomials)
  - The general solution for a(k) is obtained by taking each root of the characteristic polynomial, raising it to the kth power, and multiplying by a constant
  - The particular solution is obtained by solving for constants for which the initial conditions are satisfied

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Rules for Higher Order Systems with Polynomial Term

- The general solution a(k) is the sum of two parts:
  - Part 1 has the form of the general solution to the corresponding homogeneous dynamic system
  - Part 2 is a polynomial term
- Part 1: This part is a sum of terms
  - For each single root r_i there is a term $c_i r_i^k$
  - For each multiple root r_i there is a polynomial times $c_i r_i^k$ where the polynomial is degree 1 less than the multiplicity of the root
- Part 2: works the same as polynomial terms for first and second order systems
  - Depends on whether roots of characteristic equation are equal to 1
  - Each root equal to 1 multiplies Part 2 by a factor of k
- Compare these rules with the rules for first and second order systems
- How would they generalize to higher order systems?

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Rules for Systems with Exponential Term

- The general solution a(k) is the sum of two parts:
    - Part 1 has the form of the general solution to the corresponding homogeneous dynamic system
    - Part 2 is an exponential term b_3p^n
- Part 1: This part is a sum of terms
    - For each single root r_i there is a term $c_ir_i^k$
    - For each multiple root r_i there is a polynomial times $c_ir_i^k$ where the polynomial is degree 1 less than the multiplicity of the root
- Part 2: The form depends on whether p is a root of the characteristic equation
    - If none of the roots is equal to p then it is equal to c_3p^k
    - When a root is equal to p then a corresponding term is added to the polynomial for that root in part 1
- Compare these rules with the rules for first-order systems
- How would they generalize to higher order systems?

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Thought Questions

- **Describe some problems that can be modeled using second and higher order dynamic systems**
- **How do you find the solution to a higher order dynamic system?**
- **How do you find equilibria of higher order systems? How do you determine stability?**
- **Compare first and second order linear dynamic systems in terms of:**
  - **Solutions**
  - **Equilibrium points**

  How are they similar? Different?

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